

RADII OF STARLIKENESS AND CONVEXITY OF A CROSS-PRODUCT OF BESSEL FUNCTIONS

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ABSTRACT. In this paper some geometric properties of the normalized forms of the cross-product and product of Bessel and modified Bessel functions of the first kind are studied. For the cross-product and the product three different normalization are investigated and for each of the six functions the radii of starlikeness and convexity are precisely determined by using their Hadamard factorization. Necessary and sufficient conditions are also given for the parameters such that the six normalized functions are starlike in the open unit disk, however the convex case is open for further research. The characterization of entire functions from the Laguerre-Pólya class via hyperbolic polynomials play an important role in this paper. Moreover, the interlacing properties of the zeros of the cross-product and product of Bessel functions and their derivatives are also useful in the proof of the main results.

1. Introduction and statements of the Main Results

Let \mathbb{D}_r be the open disk $\{z \in \mathbb{C} : |z| < r\}$, where $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. As usual, with \mathcal{A} we denote the class of analytic functions $f : \mathbb{D}_r \rightarrow \mathbb{C}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Let us denote by \mathcal{S} the class of functions belonging to \mathcal{A} which are univalent in \mathbb{D}_r and let $\mathcal{S}^*(\alpha)$ be the subclass of \mathcal{S} consisting of functions which are starlike of order α in \mathbb{D}_r , where $0 \leq \alpha < 1$. The analytic characterization of this class of functions is

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{S} : \operatorname{Re} \left(\frac{zf'(z)}{f'(z)} \right) > \alpha \text{ for all } z \in \mathbb{D}_r \right\},$$

and we adopt the convention $\mathcal{S}^* = \mathcal{S}^*(0)$. The real number

$$r_\alpha^*(f) = \sup \left\{ r > 0 : \operatorname{Re} \left(\frac{zf'(z)}{f'(z)} \right) > \alpha \text{ for all } z \in \mathbb{D}_r \right\},$$

is called the radius of starlikeness of order α of the function f . It is worth to mention that $r^*(f) = r_0^*(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^*(f)})$ is a starlike domain with respect to the origin. Also, let $\mathcal{K}(\beta)$ be the subclass of \mathcal{S} consisting of functions which are convex of order β in \mathbb{D}_r , where $0 \leq \beta < 1$. The well-known analytic characterization of this class of functions is

$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{S} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta \text{ for all } z \in \mathbb{D}_r \right\},$$

and for $\beta = 0$ it reduces to the class \mathcal{K} of convex functions. The real number

$$r_\beta^c(f) = \sup \left\{ r > 0 : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta \text{ for all } z \in \mathbb{D}_r \right\},$$

is called the radius of convexity of order β of the function f . Note that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c(f)})$ is a convex domain in \mathbb{C} .

Now, let us consider the Bessel and modified Bessel functions of the first kind (see [14]), which are denoted by J_ν and I_ν , respectively. Moreover, consider the cross-product of Bessel and modified Bessel functions of the first kind

$$z \mapsto \Phi_\nu(z) = J_\nu(z)I'_\nu(z) - J'_\nu(z)I_\nu(z)$$

and the product function

$$z \mapsto \Pi_\nu(z) = J_\nu(z)I_\nu(z).$$

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Motivated by their appearance as eigenvalues in the clamped plate problem for the ball, Ashbaugh and Benguria have conjectured that the positive zeros of the function Φ_ν increase with ν on $[-1/2, \infty)$, see [2] for more details. Lorch [13] verified this conjecture and presented some other properties of the zeros of the function Φ_ν . The authors in [1] modified the proof made by Lorch to $\nu \in (-1, \infty)$, and deduced necessary and sufficient conditions for the close-to-convexity of a normalized form of the functions Φ_ν , Π_ν and their derivatives. Moreover, very recently in [6] Baricz et al. were interested on the monotonicity patterns for the cross-product of Bessel and modified Bessel functions, by showing for example that the positive zeros of the cross-product and of the Dini function $z \mapsto (1 - \nu)J_\nu(z) + zJ'_\nu(z)$ are interlacing. In [6] one of the key tools in the proofs of the main results it was the fact that the zeros of the cross-product are increasing with ν . Motivated by the above results, in this paper by using among others the above monotonicity property of the zeros of the cross-product our aim is to determine the radii of starlikeness and convexity of the normalized forms of the functions Φ_ν and Π_ν . It is worth to mention that some similar results were obtained recently in the papers [4, 9], in which the authors have studied the radii of starlikeness and convexity of q -Bessel functions as well as for Lommel and Struve functions of the first kind. We also mention that by using the corresponding recurrence relations for the Bessel and modified Bessel functions of the first kind, then the cross-product can be rewritten as

$$\Phi_\nu(z) = J_{\nu+1}(z)I_\nu(z) + J_\nu(z)I_{\nu+1}(z).$$

Moreover, we know that if $z \in \mathbb{C}$ and $\nu \in \mathbb{C}$ such that $\nu \neq -1, -2, \dots$ then the functions Φ_ν and Π_ν can be written as follows (see [17, p. 148] and [1]):

$$(1.1) \quad \Phi_\nu(z) = 2 \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2\nu+4n+1}}{n! \Gamma(\nu+n+1) \Gamma(\nu+2n+2)},$$

and

$$(1.2) \quad \Pi_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2\nu+4n}}{n! \Gamma(\nu+n+1) \Gamma(\nu+2n+1)}.$$

Since neither Φ_ν , nor Π_ν belongs to \mathcal{A} , first we perform some natural normalizations. For $\nu > -1$ we define three functions originating from Φ_ν :

$$f_\nu(z) = [2^{2\nu} \Gamma(\nu+1) \Gamma(\nu+2) \Phi_\nu(z)]^{\frac{1}{2\nu+1}}, \quad \nu \neq -\frac{1}{2},$$

$$g_\nu(z) = 2^{2\nu} z^{-2\nu} \Gamma(\nu+1) \Gamma(\nu+2) \Phi_\nu(z)$$

and

$$h_\nu(z) = 2^{2\nu} z^{-\frac{\nu}{2} + \frac{3}{4}} \Gamma(\nu+1) \Gamma(\nu+2) \Phi_\nu(\sqrt[4]{z}).$$

Similarly, we associate with Π_ν the functions

$$u_\nu(z) = [2^{2\nu} \Gamma^2(\nu+1) \Pi_\nu(z)]^{\frac{1}{2\nu}}, \quad \nu \neq 0,$$

$$v_\nu(z) = 2^{2\nu} z^{-2\nu+1} \Gamma^2(\nu+1) \Pi_\nu(z)$$

and

$$w_\nu(z) = 2^{2\nu} z^{-\frac{\nu}{2}+1} \Gamma^2(\nu+1) \Pi_\nu(\sqrt[4]{z}).$$

Clearly the functions $f_\nu, g_\nu, h_\nu, u_\nu, v_\nu$ and w_ν belong to the class \mathcal{A} . The main results in the present paper concern the exact values of the radii of starlikeness and convexity for these six functions. The paper is organized as follows: this section contains the main results of the paper and also some open problems between the second and third set of main results. Section 2 is devoted to present some preliminary results, which may be of independent interest, while section 3 contains the proofs of the main results.

The first main results we establish concern the radii of starlikeness of Φ_ν and Π_ν , and read as follows.

Theorem 1. *Let $0 \leq \alpha < 1$. The following statements hold:*

- a) *If $\nu > -\frac{1}{2}$, then $r_\alpha^*(f_\nu) = x_{\nu, \alpha, 1}$ where $x_{\nu, \alpha, 1}$ is the smallest positive root of the equation*

$$r \Phi'_\nu(r) - \alpha(2\nu+1) \Phi_\nu(r) = 0.$$

Moreover, if $\nu \in (-1, -1/2)$ then $r_\alpha^(f_\nu) = x_{\nu, \alpha}$ where $x_{\nu, \alpha}$ is the unique root of the equation*

$$i^{1/2} r \Phi'_\nu(i^{1/2} r) - \alpha(2\nu+1) \Phi_\nu(i^{1/2} r) = 0.$$

- b) *If $\nu > -1$, then $r_\alpha^*(g_\nu) = y_{\nu, \alpha, 1}$ where $y_{\nu, \alpha, 1}$ is the smallest positive root of the equation*

$$r \Phi'_\nu(r) - (\alpha + 2\nu) \Phi_\nu(r) = 0.$$

c) If $\nu > -1$, then $r_\alpha^*(h_\nu) = z_{\nu,\alpha,1}$ where $z_{\nu,\alpha,1}$ is the smallest positive root of the equation

$$r^{1/4}\Phi'_\nu(r^{1/4}) - (4\alpha + 2\nu - 3)\Phi_\nu(r^{1/4}) = 0.$$

Theorem 2. Let $0 \leq \alpha < 1$. The following statements hold:

a) If $\nu > 0$, then $r_\alpha^*(u_\nu) = \delta_{\nu,\alpha,1}$ where $\delta_{\nu,\alpha,1}$ is the smallest positive root of the equation

$$r\Pi'_\nu(r) - 2\alpha\nu\Pi_\nu(r) = 0.$$

Moreover, if $\nu \in (-1, 0)$ then $r_\alpha^*(u_\nu) = \delta_{\nu,\alpha}$ where $\delta_{\nu,\alpha}$ is the unique root of the equation

$$i^{1/2}r\Pi'_\nu(i^{1/2}r) - 2\alpha\nu\Pi_\nu(i^{1/2}r) = 0.$$

b) If $\nu > -1$, then $r_\alpha^*(v_\nu) = \rho_{\nu,\alpha,1}$ where $\rho_{\nu,\alpha,1}$ is the smallest positive root of the equation

$$r\Pi'_\nu(r) - (\alpha + 2\nu - 1)\Pi_\nu(r) = 0.$$

c) If $\nu > -1$, then $r_\alpha^*(w_\nu) = \sigma_{\nu,\alpha,1}$ where $\sigma_{\nu,\alpha,1}$ is the smallest positive root of the equation

$$r^{1/4}\Pi'_\nu(r^{1/4}) - 2(2\alpha + \nu - 2)\Pi_\nu(r^{1/4}) = 0.$$

Our second set of results concerns the radii of convexity.

Theorem 3. Suppose that $0 \leq \alpha < 1$ and let $\gamma_{\nu,1}$ and $\gamma'_{\nu,1}$ denote the first positive zeros of Φ_ν and Φ'_ν , respectively. Then the following statements hold:

a) If $\nu > -1/2$ then the radius of convexity of order α of the function f_ν is the smallest positive root of the equation

$$1 + \frac{r\Phi''_\nu(r)}{\Phi'_\nu(r)} + \left(\frac{1}{2\nu+1} - 1\right) \frac{r\Phi'_\nu(r)}{\Phi_\nu(r)} = \alpha.$$

b) If $\nu > -1$ then the radius of convexity of order α of the function g_ν is the smallest positive root of the equation

$$-2\nu + r \frac{(1-2\nu)\Phi'_\nu(r) + r\Phi''_\nu(r)}{-2\nu\Phi_\nu(r) + r\Phi'_\nu(r)} = \alpha.$$

c) If $\nu > -1$ then the radius of convexity of order α of the function h_ν is the smallest positive root of the equation

$$3 - 2\nu + r^{\frac{1}{4}} \frac{(4-2\nu)\Phi'_\nu(r^{\frac{1}{4}}) + r^{\frac{1}{4}}\Phi''_\nu(r^{\frac{1}{4}})}{(3-2\nu)\Phi_\nu(r^{\frac{1}{4}}) + r^{\frac{1}{4}}\Phi'_\nu(r^{\frac{1}{4}})} = 4\alpha.$$

Moreover, we have the inequalities $r_\alpha^c(f_\nu) < \gamma'_{\nu,1} < \gamma_{\nu,1}$, $r_\alpha^c(g_\nu) < \zeta_{\nu,1} < \gamma_{\nu,1}$, and $r_\alpha^c(h_\nu) < \xi_{\nu,1} < \gamma_{\nu,1}$ where $\zeta_{\nu,1}$ and $\xi_{\nu,1}$ are the first positive zeros of $z \mapsto z\Phi'_\nu(z) - 2\nu\Phi_\nu(z)$ and $z \mapsto z\Phi'_\nu(z) - (2\nu-3)\Phi_\nu(z)$.

Theorem 4. Suppose that $0 \leq \alpha < 1$. Let $j_{\nu,1}$ and $j'_{\nu,1}$ denote the first positive zeros of J_ν and J'_ν , respectively. Then the following statements hold:

a) If $\nu > 0$ then the radius of convexity of order α of the function u_ν is the smallest positive root of the equation

$$1 + \frac{r\Pi''_\nu(r)}{\Pi'_\nu(r)} + \left(\frac{1}{2\nu} - 1\right) \frac{r\Pi'_\nu(r)}{\Pi_\nu(r)} = \alpha.$$

b) If $\nu > -1$ then the radius of convexity of order α of the function v_ν is the smallest positive root of the equation

$$1 - 2\nu + r \frac{(2-2\nu)\Pi'_\nu(r) + r\Pi''_\nu(r)}{(1-2\nu)\Pi_\nu(r) + r\Pi'_\nu(r)} = \alpha.$$

c) If $\nu > -1$ then the radius of convexity of order α of the function w_ν is the smallest positive root of the equation

$$4 - 2\nu + r^{\frac{1}{4}} \frac{(5-2\nu)\Pi'_\nu(r^{\frac{1}{4}}) + r^{\frac{1}{4}}\Pi''_\nu(r^{\frac{1}{4}})}{(4-2\nu)\Pi_\nu(r^{\frac{1}{4}}) + r^{\frac{1}{4}}\Pi'_\nu(r^{\frac{1}{4}})} = 4\alpha.$$

Moreover, we have the inequalities $r_\alpha^c(u_\nu) < j'_{\nu,1} < j_{\nu,1}$, $r_\alpha^c(v_\nu) < \vartheta_{\nu,1} < j_{\nu,1}$, and $r_\alpha^c(w_\nu) < \omega_{\nu,1} < j_{\nu,1}$ where $\vartheta_{\nu,1}$ and $\omega_{\nu,1}$ are the first positive zeros of $z \mapsto z\Pi'_\nu(z) - (2\nu-1)\Pi_\nu(z)$ and $z \mapsto z\Pi'_\nu(z) - (2\nu-4)\Pi_\nu(z)$.

Our third set of results concerns the necessary and sufficient condition on starlikeness with respect to the origin for the six normalized functions of the cross-product and product of Bessel and modified Bessel functions of the first kind. Fig. 1 contains the graph of the left-hand sides of the equations (1.3), (1.4), (1.5), (1.6), (1.7) and (1.8) in the case when $\alpha = 0$. As we can see in the proof of these results the key tools are the monotonicity properties of the zeros of the cross-product and product with respect to the order. However, these properties seem to be not enough in order to deduce necessary and sufficient conditions on convexity, and in our opinion some Watson type derivative formulas are needed for the zeros of Φ_ν , Π_ν and their derivatives. We note that Watson formulas for the zeros of J_ν and J'_ν were recently applied fruitfully by Baricz and Szász [7] in order to deduce necessary and sufficient conditions for the parameter ν such that some normalized forms of J_ν map the open unit disk into a convex domain. Motivated by the results from [7], it would be interesting to see the counterparts of the next results for the convex case. We also mention that the particular cases of part **c** of the next theorems when $\alpha = 0$ have been deduced also in [1], however, the general case of the starlikeness of order α has not been considered there. It would be also of interest to see if the next results imply some close-to-convexity results on the derivatives of the normalized functions f_ν , g_ν , h_ν , u_ν , v_ν and w_ν , similarly as it was proved in [1] for h_ν and w_ν .

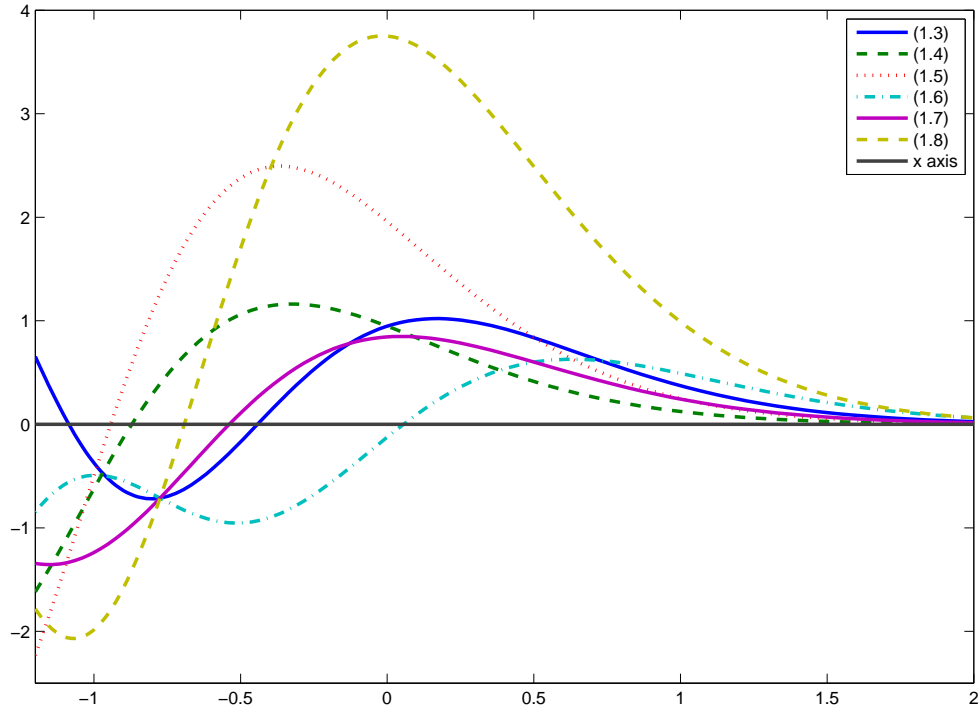


FIGURE 1. The graph of the left-hand sides of the equations (1.3), (1.4), (1.5), (1.6), (1.7) and (1.8) in the case when $\alpha = 0$.

Theorem 5. Suppose that $0 \leq \alpha < 1$. Then the following statements hold:

- a) The function f_ν is starlike of order α in \mathbb{D} if and only if $\nu \geq \nu_\alpha^*(f_\nu)$, where $\nu_\alpha^*(f_\nu)$ is the unique root of the equation

$$(1.3) \quad 2J_\nu(1)I_\nu(1) - (\alpha(2\nu + 1) + 1)(J_{\nu+1}(1)I_\nu(1) + J_\nu(1)I_{\nu+1}(1)) = 0.$$

In particular, f_ν is starlike in \mathbb{D} if and only if $\nu \geq \nu_0^*(f_\nu) \simeq -0.44\dots$, where $\nu_0^*(f_\nu)$ is the unique root of the equation (1.3) when $\alpha = 0$.

- b) The function g_ν is starlike of order α in \mathbb{D} if and only if $\nu \geq \nu_\alpha^*(g_\nu)$, where $\nu_\alpha^*(g_\nu)$ is the unique root of the equation

$$(1.4) \quad 2J_\nu(1)I_\nu(1) - (\alpha + 2\nu + 1)(J_{\nu+1}(1)I_\nu(1) + J_\nu(1)I_{\nu+1}(1)) = 0.$$

In particular, g_ν is starlike in \mathbb{D} if and only if $\nu \geq \nu_0^*(g_\nu) \simeq -0.87\dots$, where $\nu_0^*(g_\nu)$ is the unique root of the equation (1.4) when $\alpha = 0$.

- c) The function h_ν is starlike of order α in \mathbb{D} if and only if $\nu \geq \nu_\alpha^*(h_\nu)$, where $\nu_\alpha^*(h_\nu)$ is the unique root of the equation

$$(1.5) \quad J_\nu(1)I_\nu(1) - (2\alpha + \nu - 1)(J_{\nu+1}(1)I_\nu(1) + J_\nu(1)I_{\nu+1}(1)) = 0.$$

In particular, h_ν is starlike in \mathbb{D} if and only if $\nu \geq \nu_0^*(h_\nu) \simeq -0.94\dots$, where $\nu_0^*(h_\nu)$ is the unique root of the equation (1.5) when $\alpha = 0$.

Theorem 6. Suppose that $0 \leq \alpha < 1$. Then the following statements hold:

- a) The function u_ν is starlike of order α in \mathbb{D} if and only if $\nu \geq \nu_\alpha^*(u_\nu)$, where $\nu_\alpha^*(u_\nu)$ is the unique root of the equation

$$(1.6) \quad J_\nu(1)I_{\nu+1}(1) - J_{\nu+1}(1)I_\nu(1) + 2\nu(1 - \alpha)J_\nu(1)I_\nu(1) = 0.$$

In particular, u_ν is starlike in \mathbb{D} if and only if $\nu \geq \nu_0^*(u_\nu) \simeq 0.05\dots$, where $\nu_0^*(u_\nu)$ is the unique root of the equation (1.6) when $\alpha = 0$.

- b) The function v_ν is starlike of order α in \mathbb{D} if and only if $\nu \geq \nu_\alpha^*(v_\nu)$, where $\nu_\alpha^*(v_\nu)$ is the unique root of the equation

$$(1.7) \quad J_\nu(1)I_{\nu+1}(1) - J_{\nu+1}(1)I_\nu(1) + (1 - \alpha)J_\nu(1)I_\nu(1) = 0.$$

In particular, v_ν is starlike in \mathbb{D} if and only if $\nu \geq \nu_0^*(v_\nu) \simeq -0.53\dots$, where $\nu_0^*(v_\nu)$ is the unique root of the equation (1.7) when $\alpha = 0$.

- c) The function w_ν is starlike of order α in \mathbb{D} if and only if $\nu \geq \nu_\alpha^*(w_\nu)$, where $\nu_\alpha^*(w_\nu)$ is the unique root of the equation

$$(1.8) \quad J_\nu(1)I_{\nu+1}(1) - J_{\nu+1}(1)I_\nu(1) + 4(1 - \alpha)J_\nu(1)I_\nu(1) = 0.$$

In particular, w_ν is starlike in \mathbb{D} if and only if $\nu \geq \nu_0^*(w_\nu) \simeq -0.69\dots$, where $\nu_0^*(w_\nu)$ is the unique root of the equation (1.8) when $\alpha = 0$.

2. Preliminaries

2.1. The Hadamard factorizations of the functions Φ_ν and Π_ν .

Lemma 1. [1] If $\nu > -1$ and $z \in \mathbb{C}$ then the Hadamard's factorizations of Φ_ν and Π_ν are

$$(2.1) \quad \Phi_\nu(z) = \frac{z^{2\nu+1}}{2^{2\nu}\Gamma(\nu+1)\Gamma(\nu+2)} \prod_{n \geq 1} \left(1 - \frac{z^4}{\gamma_{\nu,n}^4}\right)$$

and

$$(2.2) \quad \Pi_\nu(z) = \frac{z^{2\nu}}{2^{2\nu}\Gamma^2(\nu+1)} \prod_{n \geq 1} \left(1 - \frac{z^4}{j_{\nu,n}^4}\right),$$

where $\gamma_{\nu,n}$ and $j_{\nu,n}$ are the n th positive zeros of the functions Φ_ν and J_ν . Moreover, the zeros $\gamma_{\nu,n}$ satisfy the interlacing inequalities $j_{\nu,n} < \gamma_{\nu,n} < j_{\nu,n+1}$ and $j_{\nu,n} < \gamma_{\nu,n} < j_{\nu+1,n}$ for $n \in \mathbb{N}$ and $\nu > -1$.

2.2. Quotients of power series.

We will also need the following result (see [10, 15]):

Lemma 2. Consider the power series $f(x) = \sum_{n \geq 0} a_n x^n$ and $g(x) = \sum_{n \geq 0} b_n x^n$, where $a_n \in \mathbb{R}$ and $b_n > 0$ for all $n \geq 0$. Suppose that both series converge on $(-r, r)$, for some $r > 0$. If the sequence $\{a_n/b_n\}_{n \geq 0}$ is increasing (decreasing), then the function $x \mapsto f(x)/g(x)$ is increasing (decreasing) too on $(0, r)$. The result remains true for the power series

$$f(x) = \sum_{n \geq 0} a_n x^{4n} \quad \text{and} \quad g(x) = \sum_{n \geq 0} b_n x^{4n}.$$

2.3. Zeros of hyperbolic polynomials and the Laguerre-Pólya class of entire functions. In this subsection we recall some necessary information about polynomials and entire functions with real zeros. An algebraic polynomial is called hyperbolic if all its zeros are real. We formulate the following specific statement that we shall need, see [3] for more details.

Lemma 3. *Let $p(x) = 1 - a_1x + a_2x^2 - a_3x^3 + \dots + (-1)^n a_n x^n = (1 - x/x_1) \dots (1 - x/x_n)$ be a hyperbolic polynomial with positive zeros $0 < x_1 \leq x_2 \leq \dots \leq x_n$, and normalized by $p(0) = 1$. Then, for any constant C , the polynomial $q(x) = Cp(x) - xp'(x)$ is hyperbolic. Moreover, the smallest zero η_1 belongs to the interval $(0, x_1)$ if and only if $C < 0$.*

By definition a real entire function ψ belongs to the Laguerre-Pólya class \mathcal{LP} if it can be represented in the form

$$\psi(x) = cx^m e^{-ax^2 + \beta x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

with $c, \beta, x_k \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N} \cup \{0\}$, $\sum x_k^{-2} < \infty$. Similarly, ϕ is said to be of type \mathcal{I} in the Laguerre-Pólya class, written $\varphi \in \mathcal{LP}\mathcal{I}$, if $\phi(x)$ or $\phi(-x)$ can be represented as

$$\phi(x) = cx^m e^{\sigma x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right),$$

with $c \in \mathbb{R}$, $\sigma \geq 0$, $m \in \mathbb{N} \cup \{0\}$, $x_k > 0$, $\sum 1/x_k < \infty$. The class \mathcal{LP} is the complement of the space of hyperbolic polynomials in the topology induced by the uniform convergence on the compact sets of the complex plane while $\mathcal{LP}\mathcal{I}$ is the complement of the hyperbolic polynomials whose zeros possess a preassigned constant sign. Given an entire function φ with the Maclaurin expansion

$$\varphi(x) = \sum_{n \geq 0} \tau_n \frac{x^n}{n!},$$

its Jensen polynomials are defined by

$$P_n(\varphi; x) = P_n(x) = \sum_{j=0}^n \binom{n}{j} \tau_j x^j.$$

The next result of Jensen [11] is a well-known characterization of functions belonging to \mathcal{LP} .

Lemma 4. *The function φ belongs to \mathcal{LP} ($\mathcal{LP}\mathcal{I}$, respectively) if and only if all the polynomials $P_n(\varphi; x)$, $n = 1, 2, \dots$, are hyperbolic (hyperbolic with zeros of equal sign). Moreover, the sequence $P_n(\varphi; z/n)$ converges locally uniformly to $\varphi(z)$.*

The following result is a key tool in the proof of main results.

Lemma 5. *If $\nu > -1$, then for $a < 1$ the function $z \mapsto (2\nu + a)\Phi_\nu(z) - z\Phi'_\nu(z)$, and for $b < 0$ the function $z \mapsto (2\nu + b)\Pi_\nu(z) - z\Pi'_\nu(z)$ can be represented in the form*

$$\Gamma(\nu + 1) \Gamma(\nu + 2) [(2\nu + a)\Phi_\nu(z) - z\Phi'_\nu(z)] = 2 \left(\frac{z}{2}\right)^{2\nu+1} \Psi_\nu(z),$$

$$\Gamma^2(\nu + 1) [(2\nu + b)\Pi_\nu(z) - z\Pi'_\nu(z)] = \left(\frac{z}{2}\right)^{2\nu} \phi_\nu(z),$$

where Ψ_ν and ϕ_ν are entire functions belonging to the Laguerre-Pólya class \mathcal{LP} . Moreover, the smallest positive zero of Ψ_ν does not exceed $\gamma_{\nu,1}$, while the smallest positive zero of ϕ_ν is less than $j_{\nu,1}$.

Proof. Suppose that $\nu > -1$. It is clear from Lemma 1 and the infinite product representation of $z \mapsto \Upsilon_\nu(z) = 2^{2\nu} \Gamma(\nu + 1) \Gamma(\nu + 2) z^{-2\nu-1} \Phi_\nu(z)$ that this function belongs to \mathcal{LP} . This implies that the function $z \mapsto \Upsilon_\nu(2z^{1/4}) = \tilde{\Upsilon}_\nu(z)$ belongs to $\mathcal{LP}\mathcal{I}$. Then it follows from Lemma 4 that its Jensen polynomials

$$P_n(\tilde{\Upsilon}_\nu; \zeta) = \sum_{k=0}^n \binom{n}{k} \frac{1}{(\nu + 1)_k (\nu + 2)_{2k}} (-\zeta)^k$$

are all hyperbolic, where $(c)_k = c(c+1)\dots(c+k-1)$ is the well-known Pochhammer symbol. However, observe that the Jensen polynomials of $z \mapsto \tilde{\Psi}_\nu(z) = \Psi_\nu(2z^{1/4})$ are simply

$$\frac{1}{4} P_n(\tilde{\Psi}_\nu; \zeta) = \frac{1}{4} (a - 1) P_n(\tilde{\Upsilon}_\nu; \zeta) - \zeta P'_n(\tilde{\Upsilon}_\nu; \zeta).$$

Lemma 3 implies that all zeros of $P_n(\tilde{\Psi}_\nu; \zeta)$ are real and positive and that the smallest one precedes the first zero of $P_n(\tilde{\Upsilon}_\nu; \zeta)$. In view of Lemma 4, the latter conclusion immediately yields that $\tilde{\Psi}_\nu \in \mathcal{LPI}$ and that its first zero is less than $\gamma_{\nu,1}$. Finally, the first part of the statement of the lemma follows after we go back from $\tilde{\Psi}_\nu$ to Ψ_ν by setting $\zeta = \frac{z^4}{16}$.

Similarly, the function $z \mapsto \Omega_\nu(z) = 2^{2\nu} \Gamma^2(\nu+1) z^{-2\nu} \Pi_\nu(z)$ belongs to the Laguerre-Pólya class of entire functions, which implies that the function $z \mapsto \Omega_\nu(2z^{1/4}) = \bar{\Omega}_\nu(z)$ belongs to \mathcal{LPI} . Then it follows from Lemma 4 that its Jensen polynomials

$$P_n(\bar{\Omega}_\nu; \zeta) = \sum_{k=0}^n \binom{n}{k} \frac{1}{(\nu+1)_k (\nu+1)_{2k}} (-\zeta)^k$$

are all hyperbolic. However, observe that the Jensen polynomials of $z \mapsto \tilde{\phi}_\nu(z) = \phi_\nu(2z^{1/4})$ are simply

$$\frac{1}{4} P_n(\tilde{\phi}_\nu; \zeta) = \frac{b}{4} P_n(\bar{\Omega}_\nu; \zeta) - \zeta P'_n(\bar{\Omega}_\nu; \zeta).$$

Lemma 3 implies that all zeros of $P_n(\tilde{\phi}_\nu; \zeta)$ are real and positive and that the smallest one precedes the first zero of $P_n(\bar{\Omega}_\nu; \zeta)$. In view of Lemma 4, the latter conclusion immediately yields that $\tilde{\phi}_\nu \in \mathcal{LPI}$ and that its first zero precedes $j_{\nu,1}$. Thus, the second part of the statement of this lemma follows after we go back from $\tilde{\phi}_\nu$ to ϕ_ν by setting $\zeta = \frac{z^4}{16}$. \square

2.4. The Hadamard factorization of the derivatives of Φ_ν and Π_ν . The following infinite product representations are the well-known Hadamard factorizations for the derivatives of Φ_ν and Π_ν .

Lemma 6. *For $z \in \mathbb{C}$ the Hadamard factorizations of Φ'_ν and Π'_ν are as follows: if $\nu > -1/2$ then*

$$(2.3) \quad \Phi'_\nu(z) = \frac{(2\nu+1) \left(\frac{z}{2}\right)^{2\nu}}{\Gamma(\nu+1) \Gamma(\nu+2)} \prod_{n \geq 1} \left(1 - \frac{z^4}{\gamma_{\nu,n}^4}\right),$$

and if $\nu > 0$ then

$$(2.4) \quad \Pi'_\nu(z) = \frac{\nu \left(\frac{z}{2}\right)^{2\nu-1}}{\Gamma^2(\nu+1)} \prod_{n \geq 1} \left(1 - \frac{z^4}{t_{\nu,n}^2}\right),$$

where $\gamma'_{\nu,n}$ and $t_{\nu,n}$ are the n th positive zeros of the functions Φ'_ν and Π'_ν , respectively. Moreover, if $\nu > -1/2$ then the zeros $\gamma_{\nu,n}$ and $\gamma'_{\nu,n}$ interlace; and if $\nu > 0$ then the zeros $j_{\nu,n}$ and $t_{\nu,n}$ interlace.

Proof. From (1.1) we have that

$$\frac{1}{2\nu+1} \Gamma(\nu+1) \Gamma(\nu+2) 2^{2\nu} z^{-2\nu} \Phi'_\nu(z) = \frac{1}{2\nu+1} \sum_{n \geq 0} \frac{(-1)^n (2\nu+4n+1)}{2^{4n} n! (\nu+1)_n (\nu+2)_{2n}} z^{4n},$$

and

$$\lim_{n \rightarrow \infty} \frac{n \log n}{4n \log 2 + \log \Gamma(n+1) + \log(\nu+1)_n + \log(\nu+2)_{2n} - \log(2\nu+4n+1)} = \frac{1}{4}.$$

Here we used $n! = \Gamma(n+1)$ and

$$\lim_{n \rightarrow \infty} \frac{\log \Gamma(bn+c)}{n \log n} = b,$$

where b and c are positive constants. In view of the formula of the growth order of entire functions [12, p. 6] we infer that the above entire function is of growth order $\rho = \frac{1}{4}$, and thus by applying Hadamard's Theorem [12, p. 26] the infinite product representation of Φ'_ν can be written indeed as in (2.3).

Since $\Upsilon_\nu(z) = 2^{2\nu} \Gamma(\nu+1) \Gamma(\nu+2) z^{-2\nu-1} \Phi_\nu(z)$ belongs to \mathcal{LP} (since the exponential factors in the infinite product are canceled because of the symmetry of the zeros $\pm \gamma_{\nu,n}$, $n \in \mathbb{N}$, with respect to the origin), it follows that it satisfies the Laguerre inequality (see [16])

$$(2.5) \quad \left(\Upsilon_\nu^{(n)}(x)\right)^2 - \left(\Upsilon_\nu^{(n-1)}(x)\right) \left(\Upsilon_\nu^{(n+1)}(x)\right) > 0,$$

where $\nu > -1/2$ and $z \in \mathbb{R}$. On the other hand, we have that

$$\begin{aligned} \Upsilon'_\nu(z) &= 2^{2\nu} \Gamma(\nu+1) \Gamma(\nu+2) z^{-2\nu-2} [z \Phi'_\nu(z) - (2\nu+1) \Phi_\nu(z)], \\ \Upsilon''_\nu(z) &= 2^{2\nu} \Gamma(\nu+1) \Gamma(\nu+2) z^{-2\nu-3} [z^2 \Phi''_\nu(z) - 2(2\nu+1) z \Phi'_\nu(z) + (2\nu+1)(2\nu+2) \Phi_\nu(z)], \end{aligned}$$

and thus the Laguerre inequality (2.5) for $n = 1$ is equivalent to

$$2^{4\nu}\Gamma^2(\nu+1)\Gamma^2(\nu+2)z^{-4\nu-4}\left[z^2(\Phi'_\nu(z))^2 - z^2\Phi_\nu(z)\Phi''_\nu(z) - (2\nu+1)(\Phi_\nu(z))^2\right] > 0.$$

This implies that

$$(2.6) \quad (\Phi'_\nu(z))^2 - \Phi_\nu(z)\Phi''_\nu(z) > \frac{(2\nu+1)(\Phi_\nu(z))^2}{z^2} > 0$$

for $\nu > -1/2$ and $z \in \mathbb{R}$, and thus $z \mapsto \Phi'_\nu(z)/\Phi_\nu(z)$ is decreasing on $(0, \infty) \setminus \{\gamma_{\nu,n} : n \in \mathbb{N}\}$. Since the zeros $\gamma_{\nu,n}$ of the function Φ_ν are real and simple¹, $\Phi'_\nu(z)$ does not vanish in $\gamma_{\nu,n}$, $n \in \mathbb{N}$. Thus, for a fixed $k \in \mathbb{N}$ the function $z \mapsto \Phi'_\nu(z)/\Phi_\nu(z)$ takes the limit ∞ when $z \searrow \gamma_{\nu,k-1}$, and the limit $-\infty$ when $z \nearrow \gamma_{\nu,k}$. Moreover, since $z \mapsto \Phi'_\nu(z)/\Phi_\nu(z)$ is decreasing on $(0, \infty) \setminus \{\gamma_{\nu,n} : n \in \mathbb{N}\}$ it results that in each interval $(\gamma_{\nu,k-1}, \gamma_{\nu,k})$ its restriction intersects the horizontal line only once, and the abscissa of this intersection point is exactly $\gamma'_{\nu,k}$. Consequently, the zeros $\gamma_{\nu,n}$ and $\gamma'_{\nu,n}$ interlace. Here we used the convention $\gamma_{\nu,0} = 0$.

By means of (2.2) we have

$$\frac{1}{2\nu}\Gamma^2(\nu+1)2^{2\nu}z^{-2\nu+1}\Pi'_\nu(z) = \frac{1}{2\nu} \sum_{n \geq 0} \frac{(-1)^n(2\nu+4n)}{2^{4n}n!(\nu+1)_n(\nu+1)_{2n}} z^{4n}$$

of which growth order can be written as

$$\lim_{n \rightarrow \infty} \frac{n \log n}{4n \log 2 + \log \Gamma(n+1) + \log(\nu+1)_n + \log(\nu+1)_{2n} - \log(2\nu+4n)} = \frac{1}{4}.$$

By Hadamard's Theorem [12, p. 26] the infinite product representation of Π'_ν is exactly as in (2.4).

Since $z \mapsto \Omega_\nu(z) = 2^{2\nu}\Gamma^2(\nu+1)z^{-2\nu}\Pi_\nu(z)$ belongs to \mathcal{LP} (since the exponential factors in the infinite product are canceled because of the symmetry of the zeros $\pm j_{\nu,n}$, $n \in \mathbb{N}$, with respect to the origin). On the other hand, we have that

$$\Omega'_\nu(z) = 2^{2\nu}\Gamma^2(\nu+1)z^{-2\nu-1}[z\Pi'_\nu(z) - 2\nu\Pi_\nu(z)],$$

$$\Omega''_\nu(z) = 2^{2\nu}\Gamma^2(\nu+1)z^{-2\nu-2}[z^2\Pi''_\nu(z) - 4\nu z\Pi'_\nu(z) + 2\nu(2\nu+1)\Pi_\nu(z)],$$

and thus the Laguerre inequality (see [16]) for $n = 1$ is equivalent to

$$2^{4\nu}\Gamma^4(\nu+1)z^{-4\nu-2}\left[z^2(\Pi'_\nu(z))^2 - z^2\Pi_\nu(z)\Pi''_\nu(z) - 2\nu(\Pi_\nu(z))^2\right] > 0.$$

This implies that

$$(\Pi'_\nu(z))^2 - \Pi_\nu(z)\Pi''_\nu(z) > \frac{2\nu(\Pi_\nu(z))^2}{z^2} > 0$$

for $\nu > 0$ and $z \in \mathbb{R}$, and thus $z \mapsto \Pi'_\nu(z)/\Pi_\nu(z)$ is decreasing on $(0, \infty) \setminus \{j_{\nu,n} : n \in \mathbb{N}\}$. Since the zeros $j_{\nu,n}$ of the function Π_ν are real and simple, $\Pi'_\nu(z)$ does not vanish in $j_{\nu,n}$, $n \in \mathbb{N}$. Thus, for a fixed $k \in \mathbb{N}$ the function $z \mapsto \Pi'_\nu(z)/\Pi_\nu(z)$ takes the limit ∞ when $z \searrow j_{\nu,k-1}$, and the limit $-\infty$ when $z \nearrow j_{\nu,k}$. Moreover, since $z \mapsto \Pi'_\nu(z)/\Pi_\nu(z)$ is decreasing on $(0, \infty) \setminus \{j_{\nu,n} : n \in \mathbb{N}\}$ it results that in each interval $(j_{\nu,k-1}, j_{\nu,k})$ its restriction intersects the horizontal line only once, and the abscissa of this intersection point is exactly $t_{\nu,k}$. Consequently, the zeros $j_{\nu,n}$ and $t_{\nu,n}$ interlace. Here we used the convention $j_{\nu,0} = 0$. \square

2.5. The Hadamard factorizations of the derivatives of g_ν , h_ν , v_ν and w_ν .

Lemma 7. *If $\nu > -1$ then the functions g'_ν , h'_ν , v'_ν and w'_ν are entire functions of order $\rho = \frac{1}{4}$. Consequently, their Hadamard factorizations for $z \in \mathbb{C}$ are of the form*

$$g'_\nu(z) = \prod_{n \geq 1} \left(1 - \frac{z^4}{\zeta_{\nu,n}^4}\right), \quad h'_\nu(z) = \prod_{n \geq 1} \left(1 - \frac{z}{\xi_{\nu,n}^4}\right)$$

and

$$v'_\nu(z) = \prod_{n \geq 1} \left(1 - \frac{z^4}{\vartheta_{\nu,n}^4}\right), \quad w'_\nu(z) = \prod_{n \geq 1} \left(1 - \frac{z}{\omega_{\nu,n}^4}\right),$$

¹If the zeros $\gamma_{\nu,n}$ would be not simple, let us suppose that are of multiplicity two for example, then we would have a contradiction with the inequality (2.6).

where $\zeta_{\nu,n}$ and $\xi_{\nu,n}$ are the n th positive zeros of $z \mapsto z\Phi'_\nu(z) - 2\nu\Phi_\nu(z)$ and $z \mapsto z\Phi'_\nu(z) - (2\nu - 3)\Phi_\nu(z)$ while $\vartheta_{\nu,n}$ and $\omega_{\nu,n}$ are the n th positive zeros of $z \mapsto z\Pi'_\nu(z) - (2\nu - 1)\Pi_\nu(z)$ and $z \mapsto z\Pi'_\nu(z) - (2\nu - 4)\Pi_\nu(z)$.

Proof. We have that

$$\begin{aligned} g'_\nu(z) &= 2^{2\nu}\Gamma(\nu+1)\Gamma(\nu+2)z^{-2\nu-1}[z\Phi'_\nu(z) - 2\nu\Phi_\nu(z)] = \sum_{n \geq 0} \frac{(-1)^n(4n+1)z^{4n}}{n!2^{4n}(\nu+1)_n(\nu+2)_{2n}}, \\ h'_\nu(z) &= 2^{2\nu-2}\Gamma(\nu+1)\Gamma(\nu+2)z^{-\frac{\nu}{2}-\frac{1}{4}}\left[z^{1/4}\Phi'_\nu(z^{1/4}) - (2\nu-3)\Phi_\nu(z^{1/4})\right] = \sum_{n \geq 0} \frac{(-1)^n(n+1)z^n}{n!2^{4n}(\nu+1)_n(\nu+2)_{2n}}, \\ v'_\nu(z) &= 2^{2\nu}\Gamma^2(\nu+1)z^{-2\nu}[z\Pi'_\nu(z) - (2\nu-1)\Pi_\nu(z)] = \sum_{n \geq 0} \frac{(-1)^n(4n+1)z^{4n}}{n!2^{4n}(\nu+1)_n(\nu+1)_{2n}}, \\ w'_\nu(z) &= 2^{2\nu-2}\Gamma^2(\nu+1)z^{-\frac{\nu}{2}}\left[z^{1/4}\Pi'_\nu(z^{1/4}) - (2\nu-4)\Pi_\nu(z^{1/4})\right] = \sum_{n \geq 0} \frac{(-1)^n(n+1)z^n}{n!2^{4n}(\nu+1)_n(\nu+1)_{2n}}, \end{aligned}$$

and since

$$\lim_{n \rightarrow \infty} \frac{n \log n}{4n \log 2 + \log \Gamma(n+1) + \log(\nu+1)_n + \log(\nu+a)_{2n} - \log(bn+1)} = \frac{1}{4},$$

for each $a, b > 0$, it follows that each of the above entire functions have growth order $1/4$ and hence their genus is zero. Moreover, we know that the zeros $\zeta_{\nu,n}$, $\xi_{\nu,n}$, $\vartheta_{\nu,n}$ and $\omega_{\nu,n}$, $n \in \mathbb{N}$, are real according to Lemma 5, and with this the rest of the proof follows by applying Hadamard's Theorem [12, p. 26]. \square

3. Proofs of the main results

Proof of Theorem 1. First we prove part **a** for $\nu > -1/2$ and parts **b** and **c** for $\nu > -1$. We need to show that for the corresponding values of ν and α the inequalities

$$(3.1) \quad \operatorname{Re} \left(\frac{zf'_\nu(z)}{f_\nu(z)} \right) > \alpha, \quad \operatorname{Re} \left(\frac{zg'_\nu(z)}{g_\nu(z)} \right) > \alpha \quad \text{and} \quad \operatorname{Re} \left(\frac{zh'_\nu(z)}{h_\nu(z)} \right) > \alpha$$

are valid for $z \in \mathbb{D}_{r_\alpha^*}(f_\nu)$, $z \in \mathbb{D}_{r_\alpha^*}(g_\nu)$ and $z \in \mathbb{D}_{r_\alpha^*}(h_\nu)$ respectively, and each of the above inequalities does not hold in larger disks. It follows from (2.1) that

$$\begin{aligned} \frac{zf'_\nu(z)}{f_\nu(z)} &= \frac{1}{2\nu+1} \frac{z\Phi'_\nu(z)}{\Phi_\nu(z)} = 1 - \frac{1}{2\nu+1} \sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4}, \\ \frac{zg'_\nu(z)}{g_\nu(z)} &= -2\nu + \frac{z\Phi'_\nu(z)}{\Phi_\nu(z)} = 1 - \sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4}, \end{aligned}$$

and

$$\frac{zh'_\nu(z)}{h_\nu(z)} = \frac{3}{4} - \frac{\nu}{2} + \frac{1}{4} \frac{z^{1/4}\Phi'_\nu(z^{1/4})}{\Phi_\nu(z^{1/4})} = 1 - \sum_{n \geq 1} \frac{z}{\gamma_{\nu,n}^4 - z}.$$

On the other hand, it is known that [7] if $z \in \mathbb{C}$ and $\beta \in \mathbb{R}$ are such that $\beta > |z|$, then

$$(3.2) \quad \frac{|z|}{\beta - |z|} \geq \operatorname{Re} \left(\frac{z}{\beta - z} \right).$$

Then the inequality

$$\frac{|z|^4}{\gamma_{\nu,n}^4 - |z|^4} \geq \operatorname{Re} \left(\frac{z^4}{\gamma_{\nu,n}^4 - z^4} \right),$$

holds for every $\nu > -1$, $n \in \mathbb{N}$ and $|z| < \gamma_{\nu,1}$. Therefore,

$$(3.3) \quad \operatorname{Re} \left(\frac{zf'_\nu(z)}{f_\nu(z)} \right) = 1 - \frac{1}{2\nu+1} \operatorname{Re} \left(\sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4} \right) \geq 1 - \frac{1}{2\nu+1} \sum_{n \geq 1} \frac{4|z|^4}{\gamma_{\nu,n}^4 - |z|^4} = \frac{|z|f'_\nu(|z|)}{f_\nu(|z|)},$$

$$(3.4) \quad \operatorname{Re} \left(\frac{zg'_\nu(z)}{g_\nu(z)} \right) = 1 - \operatorname{Re} \left(\sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4} \right) \geq 1 - \sum_{n \geq 1} \frac{4|z|^4}{\gamma_{\nu,n}^4 - |z|^4} = \frac{|z|g'_\nu(|z|)}{g_\nu(|z|)}$$

and

$$(3.5) \quad \operatorname{Re} \left(\frac{zh'_\nu(z)}{h_\nu(z)} \right) = 1 - \operatorname{Re} \left(\sum_{n \geq 1} \frac{z}{\gamma_{\nu,n}^4 - z} \right) \geq 1 - \sum_{n \geq 1} \frac{|z|}{\gamma_{\nu,n}^4 - |z|} = \frac{|z| h'_\nu(|z|)}{h_\nu(|z|)},$$

where equalities are attained only when $z = |z| = r$. The above inequalities and the minimum principle for harmonic functions imply that the corresponding inequalities in (3.1) hold if and only if $|z| < x_{\nu,\alpha,1}$, $|z| < y_{\nu,\alpha,1}$ and $|z| < z_{\nu,\alpha,1}$, respectively, where $x_{\nu,\alpha,1}$, $y_{\nu,\alpha,1}$ and $z_{\nu,\alpha,1}$ are the smallest positive roots of the equations

$$rf'_\nu(r)/f_\nu(r) = \alpha, \quad rg'_\nu(r)/g_\nu(r) = \alpha, \quad rh'_\nu(r)/h_\nu(r) = \alpha.$$

Since their solutions coincide with the zeros of the functions

$$\begin{aligned} r &\mapsto r\Phi'_\nu(r) - \alpha(2\nu+1)\Phi_\nu(r), \quad r \mapsto r\Phi'_\nu(r) - (\alpha+2\nu)\Phi_\nu(r), \\ r &\mapsto r^{1/4}\Phi'_\nu(r^{1/4}) - (4\alpha+2\nu-3)\Phi_\nu(r^{1/4}), \end{aligned}$$

the result we need follows from Lemma 5 by taking instead of a the values $2\nu(\alpha-1)+\alpha$, α and $4\alpha-3$, respectively. In other words, Lemma 5 shows that all the zeros of the above three functions are real and their first positive zeros do not exceed the first positive zero $\gamma_{\nu,1}$. This guarantees that the above inequalities hold. This completes the proof of part **a** when $\nu > -1/2$, and parts **b** and **c** when $\nu > -1$.

Now, to prove the statement for part **a** when $\nu \in (-1, -\frac{1}{2})$ we observe that the counterpart of (3.2) is

$$(3.6) \quad \operatorname{Re} \left(\frac{z}{\beta - z} \right) \geq \frac{-|z|}{\beta + |z|},$$

and it holds for all $z \in \mathbb{C}$ and $\beta \in \mathbb{R}$ such that $\beta > |z|$ (see [5]). From (3.6), we obtain the inequality

$$\operatorname{Re} \left(\frac{z^4}{\gamma_{\nu,n}^4 - z^4} \right) \geq \frac{-|z|^4}{\gamma_{\nu,n}^4 + |z|^4},$$

which holds for all $\nu > -1$, $n \in \mathbb{N}$ and $|z| < \gamma_{\nu,1}$ and it implies that

$$\operatorname{Re} \left(\frac{zf'_\nu(z)}{f_\nu(z)} \right) = 1 - \frac{1}{2\nu+1} \operatorname{Re} \left(\sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4} \right) \geq 1 + \frac{1}{2\nu+1} \sum_{n \geq 1} \frac{4|z|^4}{\gamma_{\nu,n}^4 + |z|^4} = \frac{i^{1/2}|z|f'_\nu(i^{1/2}|z|)}{f_\nu(i^{1/2}|z|)}.$$

In this case equality is attained if $z = i^{1/2}|z| = i^{1/2}r$. Moreover, the above inequality implies that

$$\operatorname{Re} \left(\frac{zf'_\nu(z)}{f_\nu(z)} \right) > \alpha$$

if and only if $|z| < x_{\nu,\alpha}$, where $x_{\nu,\alpha}$ denotes the smallest positive root of $i^{1/2}rf'_\nu(i^{1/2}r)/f_\nu(i^{1/2}r) = \alpha$, which is equivalent to

$$i^{1/2}r\Phi'_\nu(i^{1/2}r) - \alpha(2\nu+1)\Phi_\nu(i^{1/2}r) = 0, \quad \text{for } \nu \in \left(-1, -\frac{1}{2}\right).$$

It follows from Lemma 5 that the first positive zero of $z \mapsto i^{1/2}z\Phi'_\nu(i^{1/2}z) - \alpha(2\nu+1)\Phi_\nu(i^{1/2}z)$ does not exceed $\gamma_{\nu,1}$ which guarantees that the above inequalities are valid. All we need to prove is that the above function has actually only one zero in $(0, \infty)$. Observe that, according to Lemma 2, the function

$$r \mapsto \frac{i^{1/2}r\Phi'_\nu(i^{1/2}r)}{\Phi_\nu(i^{1/2}r)} = \sum_{n \geq 0} \frac{(4n+2\nu+1) \left(\frac{r}{2}\right)^{4n}}{n! \Gamma(\nu+n+1) \Gamma(\nu+2n+2)} \bigg/ \sum_{n \geq 0} \frac{\left(\frac{r}{2}\right)^{4n}}{n! \Gamma(\nu+n+1) \Gamma(\nu+2n+2)}$$

is increasing on $(0, \infty)$ as a quotient of two power series whose positive coefficients form the increasing “quotient sequence” $\{4n+2\nu+1\}_{n \geq 0}$. On the other hand, the above function tends to $2\nu+1$ when $r \rightarrow 0$, so that its graph can intersect the horizontal line $y = \alpha(2\nu+1) > 2\nu+1$ only once. This completes the proof of part **a** of the theorem when $\nu \in (-1, -\frac{1}{2})$. \square

Proof of Theorem 2. The proof of Theorem 2 is analogous to the proof of Theorem 1. First we prove part **a** for $\nu > 0$ and parts **b** and **c** for $\nu > -1$. From (2.2) we have

$$\frac{zu'_\nu(z)}{u_\nu(z)} = \frac{1}{2\nu} \frac{z\Pi'_\nu(z)}{\Pi_\nu(z)} = 1 - \frac{1}{2\nu} \sum_{n \geq 1} \frac{4z^4}{j_{\nu,n}^4 - z^4},$$

$$\frac{zv'_\nu(z)}{v_\nu(z)} = 1 - 2\nu + \frac{z\Pi'_\nu(z)}{\Pi_\nu(z)} = 1 - \sum_{n \geq 1} \frac{4z^4}{j_{\nu,n}^4 - z^4},$$

and

$$\frac{zw'_\nu(z)}{w_\nu(z)} = 1 - \frac{\nu}{2} + \frac{1}{4} \frac{z^{1/4}\Pi'_\nu(z^{1/4})}{\Pi_\nu(z^{1/4})} = 1 - \sum_{n \geq 1} \frac{z}{j_{\nu,n}^4 - z}.$$

On the other hand, by means of (3.2) we have the inequality

$$\frac{|z|^4}{j_{\nu,n}^4 - |z|^4} \geq \operatorname{Re} \left(\frac{z^4}{j_{\nu,n}^4 - z^4} \right),$$

for every $\nu > -1$, $n \in \mathbb{N}$ and $|z| < j_{\nu,1}$. Therefore,

$$(3.7) \quad \operatorname{Re} \left(\frac{zu'_\nu(z)}{u_\nu(z)} \right) \geq 1 - \frac{1}{2\nu} \sum_{n \geq 1} \frac{4|z|^4}{j_{\nu,n}^4 - |z|^4} = \frac{|z|u'_\nu(|z|)}{u_\nu(|z|)},$$

$$(3.8) \quad \operatorname{Re} \left(\frac{zv'_\nu(z)}{v_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{4|z|^4}{j_{\nu,n}^4 - |z|^4} = \frac{|z|v'_\nu(|z|)}{v_\nu(|z|)}$$

and

$$(3.9) \quad \operatorname{Re} \left(\frac{zw'_\nu(z)}{w_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{|z|}{j_{\nu,n}^4 - |z|} = \frac{|z|w'_\nu(|z|)}{w_\nu(|z|)},$$

where equalities are attained only when $z = |z| = r$. The above inequalities and the minimum principle for harmonic functions imply that the inequalities

$$\operatorname{Re} \left(\frac{zu'_\nu(z)}{u_\nu(z)} \right) > \alpha, \quad \operatorname{Re} \left(\frac{zv'_\nu(z)}{v_\nu(z)} \right) > \alpha \quad \text{and} \quad \operatorname{Re} \left(\frac{zw'_\nu(z)}{w_\nu(z)} \right) > \alpha$$

hold if and only if $|z| < \delta_{\nu,\alpha,1}$, $|z| < \rho_{\nu,\alpha,1}$ and $|z| < \sigma_{\nu,\alpha,1}$, respectively, where $\delta_{\nu,\alpha,1}$, $\rho_{\nu,\alpha,1}$ and $\sigma_{\nu,\alpha,1}$ are the smallest positive roots of the equations

$$ru'_\nu(r)/u_\nu(r) = \alpha, \quad rv'_\nu(r)/v_\nu(r) = \alpha, \quad rw'_\nu(r)/w_\nu(r) = \alpha.$$

Since their solutions coincide with the zeros of the functions

$$r \mapsto r\Pi'_\nu(r) - 2\alpha\nu\Pi_\nu(r), \quad r \mapsto r\Pi'_\nu(r) - (\alpha + 2\nu - 1)\Pi_\nu(r), \\ r \mapsto r^{1/4}\Pi'_\nu(r^{1/4}) - 2(2\alpha + \nu - 2)\Pi_\nu(r^{1/4}),$$

the result we need follows from Lemma 5 by taking instead of b the values $2\nu(\alpha-1)$, $\alpha-1$ and $4(\alpha-1)$, respectively. In other words, Lemma 5 shows that all the zeros of the above three functions are real and their first positive zeros do not exceed the first positive zero $j_{\nu,1}$. This guarantees that the above inequalities hold. This completes the proof of part **a** when $\nu > 0$, and parts **b** and **c** when $\nu > -1$.

Now, to prove the statement for part **a** when $\nu \in (-1, 0)$. From (3.6), we obtain the inequality

$$\operatorname{Re} \left(\frac{z^4}{j_{\nu,n}^4 - z^4} \right) \geq \frac{-|z|^4}{j_{\nu,n}^4 + |z|^4},$$

which holds for all $\nu > -1$, $n \in \mathbb{N}$ and $|z| < j_{\nu,1}$ and it implies that

$$\operatorname{Re} \left(\frac{zu'_\nu(z)}{u_\nu(z)} \right) \geq 1 + \frac{1}{2\nu} \sum_{n \geq 1} \frac{4|z|^4}{j_{\nu,n}^4 + |z|^4} = \frac{i^{1/2}|z|u'_\nu(i^{1/2}|z|)}{u_\nu(i^{1/2}|z|)}.$$

In this case equality is attained if $z = i^{1/2}|z| = i^{1/2}r$. Moreover, the above inequality implies that

$$\operatorname{Re} \left(\frac{zu'_\nu(z)}{u_\nu(z)} \right) > \alpha$$

if and only if $|z| < \delta_{\nu,\alpha}$, where $\delta_{\nu,\alpha}$ denotes the smallest positive root of $i^{1/2}ru'_\nu(i^{1/2}r)/u_\nu(i^{1/2}r) = \alpha$, which is equivalent to

$$i^{1/2}r\Pi'_\nu(i^{1/2}r) - 2\alpha\nu\Pi_\nu(i^{1/2}r) = 0, \quad \text{for } \nu \in (-1, 0).$$

It follows from Lemma 5 that the first positive zero of $z \mapsto i^{1/2} z \Pi'_\nu(i^{1/2} z) - 2\alpha\nu \Pi_\nu(i^{1/2} z)$ does not exceed $j_{\nu,1}$ which guarantees that the above inequalities are valid. All we need to prove is that the above function has actually only one zero in $(0, \infty)$. Observe that, according to Lemma 2, the function

$$r \mapsto \frac{i^{1/2} r \Pi'_\nu(i^{1/2} r)}{\Pi_\nu(i^{1/2} r)} = \sum_{n \geq 0} \frac{(4n+2\nu) \left(\frac{r}{2}\right)^{4n}}{n! \Gamma(\nu+n+1) \Gamma(\nu+2n+1)} \Bigg/ \sum_{n \geq 0} \frac{\left(\frac{r}{2}\right)^{4n}}{n! \Gamma(\nu+n+1) \Gamma(\nu+2n+1)}$$

is increasing on $(0, \infty)$ as a quotient of two power series whose positive coefficients form the increasing “quotient sequence” $\{4n+2\nu\}_{n \geq 0}$. On the other hand, the above function tends to 2ν when $r \rightarrow 0$, so that its graph can intersect the horizontal line $y = 2\alpha\nu > 2\nu$ only once. This completes the proof of part **a** of the theorem when $\nu \in (-1, 0)$. \square

Proof of Theorem 3. a) Observe that

$$1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 + \frac{z\Phi''_\nu(z)}{\Phi'_\nu(z)} + \left(\frac{1}{2\nu+1} - 1\right) \frac{z\Phi'_\nu(z)}{\Phi_\nu(z)}, \quad \nu \neq -\frac{1}{2}.$$

By means of (2.1) and (2.3) we have

$$(3.10) \quad \frac{z\Phi'_\nu(z)}{\Phi_\nu(z)} = 2\nu + 1 - \sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4}, \quad 1 + \frac{z\Phi''_\nu(z)}{\Phi'_\nu(z)} = 2\nu + 1 - \sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4},$$

and it follows that

$$1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 - \left(\frac{1}{2\nu+1} - 1\right) \sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4} - \sum_{n \geq 1} \frac{4z^4}{\gamma_{\nu,n}^4 - z^4}, \quad \nu \neq -\frac{1}{2}.$$

Now, suppose that $\nu \in (-\frac{1}{2}, 0]$. By using (3.2), we obtain for all $z \in \mathbb{D}_{\gamma'_{\nu,1}}$ the inequality

$$\operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) \geq 1 - \left(\frac{1}{2\nu+1} - 1\right) \sum_{n \geq 1} \frac{4r^4}{\gamma_{\nu,n}^4 - r^4} - \sum_{n \geq 1} \frac{4r^4}{\gamma_{\nu,n}^4 - r^4},$$

where $|z| = r$. Moreover, observe that if we use the inequality [7, Lemma 2.1]

$$(3.11) \quad \lambda \operatorname{Re} \left(\frac{z}{a-z} \right) - \operatorname{Re} \left(\frac{z}{b-z} \right) \geq \lambda \frac{|z|}{a-|z|} - \frac{|z|}{b-|z|},$$

where $a > b > 0$, $\lambda \in [0, 1]$ and $z \in \mathbb{C}$ such that $|z| < b$, then we get that the above inequality is also valid when $\nu > 0$. Here we used that the zeros $\gamma_{\nu,n}$ and $\gamma'_{\nu,n}$ interlace. Thus, for $r \in (0, \gamma'_{\nu,1})$ we have

$$\inf_{z \in \mathbb{D}_r} \left\{ \operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) \right\} = 1 + \frac{rf''_\nu(r)}{f'_\nu(r)}.$$

On the other hand, the function $F_\nu : (0, \gamma'_{\nu,1}) \rightarrow \mathbb{R}$, defined by

$$F_\nu(r) = 1 + \frac{rf''_\nu(r)}{f'_\nu(r)},$$

is strictly decreasing for all $\nu > -\frac{1}{2}$. Namely, we have

$$\begin{aligned} F'_\nu(r) &= - \left(\frac{1}{2\nu+1} - 1 \right) \sum_{n \geq 1} \frac{16r^3 \gamma_{\nu,n}^4}{(\gamma_{\nu,n}^4 - r^4)^2} - \sum_{n \geq 1} \frac{16r^3 \gamma_{\nu,n}^4}{(\gamma_{\nu,n}^4 - r^4)^2} \\ &< \sum_{n \geq 1} \frac{16r^3 \gamma_{\nu,n}^4}{(\gamma_{\nu,n}^4 - r^4)^2} - \sum_{n \geq 1} \frac{16r^3 \gamma_{\nu,n}^4}{(\gamma_{\nu,n}^4 - r^4)^2} < 0 \end{aligned}$$

for $\nu > -\frac{1}{2}$ and $r \in (0, \gamma'_{\nu,1})$. Here we used again that the zeros $\gamma_{\nu,n}$ and $\gamma'_{\nu,n}$ interlace, and for all $n \in \mathbb{N}$, $\nu > -\frac{1}{2}$ and $r < \sqrt{\gamma_{\nu,n} \gamma'_{\nu,n}}$ we have that

$$\gamma_{\nu,n}^4 (\gamma_{\nu,n}^4 - r^4)^2 < \gamma_{\nu,n}^4 (\gamma_{\nu,n}^4 - r^4)^2.$$

Now, since $\lim_{r \searrow 0} F_\nu(r) = 1 > \alpha$ and $\lim_{r \nearrow \gamma'_{\nu,1}} F_\nu(r) = -\infty$, in view of the minimum principle for harmonic functions it follows that for $\nu > -\frac{1}{2}$ and $z \in \mathbb{D}_{r_1}$ we have

$$(3.12) \quad \operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) > \alpha$$

if and only if r_1 is the unique root of

$$1 + \frac{rf''_\nu(r)}{f'_\nu(r)} = \alpha$$

situated in $(0, \gamma'_{\nu,1})$. This completes the proof of part **a** of our theorem when $\nu > -\frac{1}{2}$.

b) In view of Lemma 7 we have that

$$1 + \frac{zg''_\nu(z)}{g'_\nu(z)} = 1 - \sum_{n \geq 1} \frac{4z^4}{\zeta_{\nu,n}^4 - z^4}.$$

By using the inequality (3.2), for all $z \in \mathbb{D}_{\zeta_{\nu,1}}$ we obtain the inequality

$$\operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{4r^4}{\zeta_{\nu,n}^4 - r^4},$$

where $|z| = r$. Thus, for $r \in (0, \zeta_{\nu,1})$ we get

$$\inf_{z \in \mathbb{D}_r} \left\{ \operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) \right\} = 1 + \frac{rg''_\nu(r)}{g'_\nu(r)}.$$

The function $G_\nu : (0, \zeta_{\nu,1}) \rightarrow \mathbb{R}$, defined by

$$G_\nu(r) = 1 + \frac{rg''_\nu(r)}{g'_\nu(r)},$$

is strictly decreasing and $\lim_{r \searrow 0} G_\nu(r) = 1 > \alpha$, $\lim_{r \nearrow \zeta_{\nu,1}} G_\nu(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_2}$ we have that

$$\operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) > \alpha$$

if and only if r_2 is the unique root of

$$1 + \frac{rg''_\nu(r)}{g'_\nu(r)} = \alpha$$

situated in $(0, \zeta_{\nu,1})$. Finally, the inequality $\zeta_{\nu,1} < \gamma_{\nu,1}$ follows from Lemma 5.

c) In view of Lemma 7 we have that

$$1 + \frac{zh''_\nu(z)}{h'_\nu(z)} = 1 - \sum_{n \geq 1} \frac{z}{\xi_{\nu,n}^4 - z}.$$

By using the inequality (3.2), for all $z \in \mathbb{D}_{\xi_{\nu,1}}$ we obtain the inequality

$$\operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{r}{\xi_{\nu,n}^4 - r},$$

where $|z| = r$. Thus, for $r \in (0, \xi_{\nu,1})$ we get

$$\inf_{z \in \mathbb{D}_r} \left\{ \operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) \right\} = 1 + \frac{rh''_\nu(r)}{h'_\nu(r)}.$$

The function $H_\nu : (0, \xi_{\nu,1}) \rightarrow \mathbb{R}$, defined by

$$H_\nu(r) = 1 + \frac{rh''_\nu(r)}{h'_\nu(r)},$$

is strictly decreasing and $\lim_{r \searrow 0} H_\nu(r) = 1 > \alpha$, $\lim_{r \nearrow \xi_{\nu,1}} H_\nu(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_3}$ we have that

$$\operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) > \alpha$$

if and only if r_3 is the unique root of

$$1 + \frac{rh''_\nu(r)}{h'_\nu(r)} = \alpha$$

situated in $(0, \xi_{\nu,1})$. Finally, the inequality $\xi_{\nu,1} < \gamma_{\nu,1}$ follows from Lemma 5. □

Proof of Theorem 4. a) By means of (2.2) and (2.4) we have

$$\begin{aligned} 1 + \frac{zu''_\nu(z)}{u'_\nu(z)} &= 1 + \frac{z \Pi''_\nu(z)}{\Pi'_\nu(z)} + \left(\frac{1}{2\nu} - 1 \right) \frac{z \Pi'_\nu(z)}{\Pi_\nu(z)}, \quad \nu \neq 0 \\ &= 1 - \left(\frac{1}{2\nu} - 1 \right) \sum_{n \geq 1} \frac{4z^4}{j_{\nu,n}^4 - z^4} - \sum_{n \geq 1} \frac{4z^4}{t_{\nu,n}^4 - z^4}. \end{aligned}$$

Now, suppose that $\nu \in (0, \frac{1}{2}]$. By using the inequality (3.2), for all $z \in \mathbb{D}_{t_{\nu,1}}$ we obtain the inequality

$$\operatorname{Re} \left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)} \right) \geq 1 - \left(\frac{1}{2\nu} - 1 \right) \sum_{n \geq 1} \frac{4r^4}{j_{\nu,n}^4 - r^4} - \sum_{n \geq 1} \frac{4r^4}{t_{\nu,n}^4 - r^4},$$

where $|z| = r$. Moreover, observe that if we use the inequality (3.11) then we get that the above inequality is also valid when $\nu > \frac{1}{2}$. Here we used that the zeros $j_{\nu,n}$ and $t_{\nu,n}$ interlace. The above inequality implies for $r \in (0, t_{\nu,1})$

$$\inf_{z \in \mathbb{D}_r} \left\{ \operatorname{Re} \left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)} \right) \right\} = 1 + \frac{ru''_\nu(r)}{u'_\nu(r)}.$$

On the other hand, the function $U_\nu : (0, t_{\nu,1}) \rightarrow \mathbb{R}$, defined by

$$U_\nu(r) = 1 + \frac{ru''_\nu(r)}{u'_\nu(r)},$$

is strictly decreasing since

$$\begin{aligned} U'_\nu(r) &= - \left(\frac{1}{2\nu} - 1 \right) \sum_{n \geq 1} \frac{16r^3 j_{\nu,n}^4}{(j_{\nu,n}^4 - r^4)^2} - \sum_{n \geq 1} \frac{16r^3 t_{\nu,n}^4}{(t_{\nu,n}^4 - r^4)^2} \\ &< \sum_{n \geq 1} \frac{16r^3 j_{\nu,n}^4}{(j_{\nu,n}^4 - r^4)^2} - \sum_{n \geq 1} \frac{16r^3 t_{\nu,n}^4}{(t_{\nu,n}^4 - r^4)^2} < 0 \end{aligned}$$

for $\nu > 0$ and $r \in (0, t_{\nu,1})$. Here we used again that the zeros $j_{\nu,n}$ and $t_{\nu,n}$ interlace for all $n \in \mathbb{N}$, $\nu > 0$ and $r < \sqrt{j_{\nu,n} t_{\nu,n}}$ we have that

$$j_{\nu,n}^4 (t_{\nu,n}^4 - r^4)^2 < t_{\nu,n}^4 (j_{\nu,n}^4 - r^4)^2.$$

Since $\lim_{r \searrow 0} U_\nu(r) = 1 > \alpha$ and $\lim_{r \nearrow t_{\nu,1}} U_\nu(r) = -\infty$, in view of the minimum principle for harmonic functions it follows that for $z \in \mathbb{D}_{r_4}$ we have

$$\operatorname{Re} \left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)} \right) > \alpha$$

if and only if r_4 is the unique root of

$$1 + \frac{ru''_\nu(r)}{u'_\nu(r)} = \alpha$$

situated in $(0, t_{\nu,1})$.

b) In view of Lemma 7 we have that

$$1 + \frac{zv''_\nu(z)}{v'_\nu(z)} = 1 - \sum_{n \geq 1} \frac{4z^4}{\vartheta_{\nu,n}^4 - z^4}.$$

By using the inequality (3.2), for all $z \in \mathbb{D}_{\vartheta_{\nu,1}}$ we obtain the inequality

$$\operatorname{Re} \left(1 + \frac{zv''_\nu(z)}{v'_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{4r^4}{\vartheta_{\nu,n}^4 - r^4},$$

where $|z| = r$. Thus, for $r \in (0, \vartheta_{\nu,1})$ we get

$$\inf_{z \in \mathbb{D}_r} \left\{ \operatorname{Re} \left(1 + \frac{zv''_\nu(z)}{v'_\nu(z)} \right) \right\} = 1 + \frac{rv''_\nu(r)}{v'_\nu(r)}.$$

On the other hand, the function $V_\nu : (0, \vartheta_{\nu,1}) \rightarrow \mathbb{R}$, defined by

$$V_\nu(r) = 1 + \frac{rv''_\nu(r)}{v'_\nu(r)},$$

is strictly decreasing and $\lim_{r \searrow 0} V_\nu(r) = 1 > \alpha$ and $\lim_{r \nearrow \vartheta_{\nu,1}} V_\nu(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_5}$ we have that

$$\operatorname{Re} \left(1 + \frac{zv''_\nu(z)}{v'_\nu(z)} \right) > \alpha$$

if and only if r_5 is the unique root of

$$1 + \frac{rv''_\nu(r)}{v'_\nu(r)} = \alpha$$

situated in $(0, \vartheta_{\nu,1})$. Finally, the inequality $\vartheta_{\nu,1} < j_{\nu,1}$ follows from Lemma 5.

c) Similarly, from Lemma 7 by using

$$1 + \frac{zw''_\nu(z)}{w'_\nu(z)} = 1 - \sum_{n \geq 1} \frac{z}{\omega_{\nu,n}^4 - z}.$$

and the inequality (3.2), we get for all $z \in \mathbb{D}_{\omega_{\nu,1}}$

$$\operatorname{Re} \left(1 + \frac{zw''_\nu(z)}{w'_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{r}{\omega_{\nu,n}^4 - r},$$

where $|z| = r$. Hence, for $r \in (0, \omega_{\nu,1})$ we obtain

$$\inf_{z \in \mathbb{D}_r} \left\{ \operatorname{Re} \left(1 + \frac{zw''_\nu(z)}{w'_\nu(z)} \right) \right\} = 1 + \frac{rw''_\nu(r)}{w'_\nu(r)}.$$

On the other hand, the function $W_\nu : (0, \omega_{\nu,1}) \rightarrow \mathbb{R}$, defined by

$$W_\nu(r) = 1 + \frac{rw''_\nu(r)}{w'_\nu(r)},$$

is strictly decreasing and $\lim_{r \searrow 0} W_\nu(r) = 1 > \alpha$ and $\lim_{r \nearrow \omega_{\nu,1}} W_\nu(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_6}$ we have that

$$\operatorname{Re} \left(1 + \frac{zw''_\nu(z)}{w'_\nu(z)} \right) > \alpha$$

if and only if r_6 is the unique root of

$$1 + \frac{rw''_\nu(r)}{w'_\nu(r)} = \alpha$$

situated in $(0, \omega_{\nu,1})$. Finally, the inequality $\omega_{\nu,1} < j_{\nu,1}$ follows from Lemma 5. \square

Proof of Theorem 5. Recall, that $\nu \mapsto \gamma_{\nu,n}$ is increasing on $(-1, \infty)$ for $n \in \mathbb{N}$ fixed, see [1, Lemma 4]. Thus, we have that $\gamma_{\nu,1} > 1$ if and only if $\nu > \nu_*$, where $\nu_* \simeq -0.97\dots$ is the unique root of $\gamma_{\nu,1} = 1$, or equivalently $\Phi_\nu(1) = 0$, or equivalently $J_{\nu+1}(1)I_\nu(1) + J_\nu(1)I_{\nu+1}(1) = 0$. Consequently, we have that for $\nu > \nu_*$ and $n \in \mathbb{N}$ all zeros $\gamma_{\nu,n}$ are outside of the unit disk \mathbb{D} . According to (3.3), (3.4) and (3.5) for $z \in \mathbb{D}$, $|z| = r$ we get

$$\operatorname{Re} \left(\frac{zf'_\nu(z)}{f_\nu(z)} \right) \geq 1 - \frac{1}{2\nu+1} \sum_{n \geq 1} \frac{4r^4}{\gamma_{\nu,n}^4 - r^4} = \frac{1}{2\nu+1} \frac{r\Phi'_\nu(r)}{\Phi_\nu(r)} = \frac{rf'_\nu(r)}{f_\nu(r)}, \quad \nu > -\frac{1}{2},$$

$$\operatorname{Re} \left(\frac{zg'_\nu(z)}{g_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{4r^4}{\gamma_{\nu,n}^4 - r^4} = -2\nu + \frac{r\Phi'_\nu(r)}{\Phi_\nu(r)} = \frac{rg'_\nu(r)}{g_\nu(r)}, \quad \nu > -1,$$

and

$$\operatorname{Re} \left(\frac{zh'_\nu(z)}{h_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{r}{\gamma_{\nu,n}^4 - r} = \frac{3}{4} - \frac{\nu}{2} + \frac{1}{4} \frac{r^{1/4}\Phi'_\nu(r^{1/4})}{\Phi_\nu(r^{1/4})} = \frac{rh'_\nu(r)}{h_\nu(r)}, \quad \nu > -1,$$

since $\gamma_{\nu,n} > \gamma_{\nu,1} > 1$ for $\nu > \nu_*$ and $n \in \mathbb{N}$. Now, we have

$$\frac{\partial}{\partial r} \left(\frac{rf'_\nu(r)}{f_\nu(r)} \right) = -\frac{16}{2\nu+1} \sum_{n \geq 1} \frac{\gamma_{\nu,n}^4 r^3}{(\gamma_{\nu,n}^4 - r^4)^2} < 0, \quad \nu > -\frac{1}{2},$$

$$\frac{\partial}{\partial r} \left(\frac{rg'_\nu(r)}{g_\nu(r)} \right) = -16 \sum_{n \geq 1} \frac{\gamma_{\nu,n}^4 r^3}{(\gamma_{\nu,n}^4 - r^4)^2} < 0, \quad \nu > -1,$$

and

$$\frac{\partial}{\partial r} \left(\frac{rh'_\nu(r)}{h_\nu(r)} \right) = - \sum_{n \geq 1} \frac{\gamma_{\nu,n}^4}{(\gamma_{\nu,n}^4 - r)^2} < 0, \quad \nu > -1.$$

These imply that the functions $r \mapsto rf'_\nu(r)/f_\nu(r)$, $r \mapsto rg'_\nu(r)/g_\nu(r)$ and $r \mapsto rh'_\nu(r)/h_\nu(r)$ are decreasing on $(0, 1) \subset (0, \gamma_{\nu,1})$. So, for $z \in \mathbb{D}$ and for the corresponding ν we have

$$\operatorname{Re} \left(\frac{zf'_\nu(z)}{f_\nu(z)} \right) \geq \frac{rf'_\nu(r)}{f_\nu(r)} \geq \frac{f'_\nu(1)}{f_\nu(1)} = 1 - \frac{1}{2\nu+1} \sum_{n \geq 1} \frac{4}{\gamma_{\nu,n}^4 - 1}, \quad \nu > -\frac{1}{2},$$

$$\operatorname{Re} \left(\frac{zg'_\nu(z)}{g_\nu(z)} \right) \geq \frac{rg'_\nu(r)}{g_\nu(r)} \geq \frac{g'_\nu(1)}{g_\nu(1)} = 1 - \sum_{n \geq 1} \frac{4}{\gamma_{\nu,n}^4 - 1}, \quad \nu > -1,$$

and

$$\operatorname{Re} \left(\frac{zh'_\nu(z)}{h_\nu(z)} \right) \geq \frac{rh'_\nu(r)}{h_\nu(r)} \geq \frac{h'_\nu(1)}{h_\nu(1)} = 1 - \sum_{n \geq 1} \frac{1}{\gamma_{\nu,n}^4 - 1}, \quad \nu > -1.$$

By using again the fact that the function $\nu \mapsto \gamma_{\nu,n}$ is increasing on $(-1, \infty)$ for all fixed $n \in \mathbb{N}$, we get that the functions $\nu \mapsto f'_\nu(1)/f_\nu(1)$, $\nu \mapsto g'_\nu(1)/g_\nu(1)$ and $\nu \mapsto h'_\nu(1)/h_\nu(1)$ are also increasing on $(-1/2, \infty)$ and $(-1, \infty)$, respectively. Therefore the following statements are valid for $0 \leq \alpha < 1$:

- $f'_\nu(1)/f_\nu(1) > \alpha$ if and only if $\nu \geq \nu_\alpha^*(f_\nu)$, where $\nu_\alpha^*(f_\nu)$ is the unique root of $f'_\nu(1) = \alpha f_\nu(1)$.
- $g'_\nu(1)/g_\nu(1) > \alpha$ if and only if $\nu \geq \nu_\alpha^*(g_\nu)$, where $\nu_\alpha^*(g_\nu)$ is the unique root of $g'_\nu(1) = \alpha g_\nu(1)$.
- $h'_\nu(1)/h_\nu(1) > \alpha$ if and only if $\nu \geq \nu_\alpha^*(h_\nu)$, where $\nu_\alpha^*(h_\nu)$ is the unique root of $h'_\nu(1) = \alpha h_\nu(1)$.

The above equations are equivalent to

$$2\Pi_\nu(1) - (\alpha(2\nu+1) + 1)\Phi_\nu(1) = 0,$$

$$2\Pi_\nu(1) - (\alpha + 2\nu + 1)\Phi_\nu(1) = 0$$

and

$$\Pi_\nu(1) - (2\alpha + \nu - 1)\Phi_\nu(1) = 0$$

which are equivalent to the equations in the statements of the theorem. This completes the proof. \square

Proof of Theorem 6. According to (3.7), (3.8) and (3.9) for $z \in \mathbb{D}$ and $|z| = r$, we obtain

$$\operatorname{Re} \left(\frac{zu'_\nu(z)}{u_\nu(z)} \right) \geq 1 - \frac{1}{2\nu} \sum_{n \geq 1} \frac{4r^4}{j_{\nu,n}^4 - r^4} = \frac{1}{2\nu} \frac{r\Pi'_\nu(r)}{\Pi_\nu(r)} = \frac{ru'_\nu(r)}{u_\nu(r)}, \quad \nu > 0,$$

$$\operatorname{Re} \left(\frac{zv'_\nu(z)}{v_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{4r^4}{j_{\nu,n}^4 - r^4} = 1 - 2\nu + \frac{r\Pi'_\nu(r)}{\Pi_\nu(r)} = \frac{rv'_\nu(r)}{v_\nu(r)}, \quad \nu > -1,$$

and

$$\operatorname{Re} \left(\frac{zw'_\nu(z)}{w_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{r}{j_{\nu,n}^4 - r} = 1 - \frac{\nu}{2} + \frac{1}{4} \frac{r^{1/4}\Pi'_\nu(r^{1/4})}{\Pi_\nu(r^{1/4})} = \frac{rw'_\nu(r)}{w_\nu(r)}, \quad \nu > -1,$$

since $j_{\nu,n} > j_{\nu,1} > 1$ for $n \in \mathbb{N}$ and $\nu > \nu^\circ \simeq -0.77\dots$, where ν° is unique root of the equation $j_{\nu,1} = 1$, see [8] for more details. Now, we get

$$\frac{\partial}{\partial r} \left(\frac{ru'_\nu(r)}{u_\nu(r)} \right) = -\frac{8}{\nu} \sum_{n \geq 1} \frac{j_{\nu,n}^4 r^3}{(j_{\nu,n}^4 - r^4)^2} < 0, \quad \nu > 0,$$

$$\frac{\partial}{\partial r} \left(\frac{rv'_\nu(r)}{v_\nu(r)} \right) = -16 \sum_{n \geq 1} \frac{j_{\nu,n}^4 r^3}{(j_{\nu,n}^4 - r^4)^2} < 0, \quad \nu > -1,$$

and

$$\frac{\partial}{\partial r} \left(\frac{rw'_\nu(r)}{w_\nu(r)} \right) = - \sum_{n \geq 1} \frac{j_{\nu,n}^4}{(j_{\nu,n}^4 - r)^2} < 0, \quad \nu > -1.$$

So, the functions $r \mapsto ru'_\nu(r)/u_\nu(r)$, $r \mapsto rv'_\nu(r)/v_\nu(r)$ and $r \mapsto rw'_\nu(r)/w_\nu(r)$ are decreasing on $(0, 1) \subset (0, j_{\nu,1})$. Hence,

$$\operatorname{Re} \left(\frac{zu'_\nu(z)}{u_\nu(z)} \right) \geq \frac{ru'_\nu(r)}{u_\nu(r)} \geq \frac{u'_\nu(1)}{u_\nu(1)} = 1 - \frac{1}{2\nu} \sum_{n \geq 1} \frac{4}{j_{\nu,n}^4 - 1}, \quad \nu > 0,$$

$$\operatorname{Re} \left(\frac{zv'_\nu(z)}{v_\nu(z)} \right) \geq \frac{rv'_\nu(r)}{v_\nu(r)} \geq \frac{v'_\nu(1)}{v_\nu(1)} = 1 - \sum_{n \geq 1} \frac{4}{j_{\nu,n}^4 - 1}, \quad \nu > -1,$$

and

$$\operatorname{Re} \left(\frac{zw'_\nu(z)}{w_\nu(z)} \right) \geq \frac{rw'_\nu(r)}{w_\nu(r)} \geq \frac{w'_\nu(1)}{w_\nu(1)} = 1 - \sum_{n \geq 1} \frac{1}{j_{\nu,n}^4 - 1}, \quad \nu > -1.$$

Since $\nu \mapsto j_{\nu,n}$ is increasing on $(-1, \infty)$ for all fixed $n \in \mathbb{N}$, the functions $\nu \mapsto u'_\nu(1)/u_\nu(1)$, $\nu \mapsto v'_\nu(1)/v_\nu(1)$ and $\nu \mapsto w'_\nu(1)/w_\nu(1)$ are also increasing on $(0, \infty)$ and $(-1, \infty)$, respectively. Therefore the following statements are true for $0 \leq \alpha < 1$:

- $u'_\nu(1)/u_\nu(1) > \alpha$ if and only if $\nu \geq \nu_\alpha^*(u_\nu)$, where $\nu_\alpha^*(u_\nu)$ is the unique root of $u'_\nu(1) = \alpha u_\nu(1)$.
- $v'_\nu(1)/v_\nu(1) > \alpha$ if and only if $\nu \geq \nu_\alpha^*(v_\nu)$, where $\nu_\alpha^*(v_\nu)$ is the unique root of $v'_\nu(1) = \alpha v_\nu(1)$.
- $w'_\nu(1)/w_\nu(1) > \alpha$ if and only if $\nu \geq \nu_\alpha^*(w_\nu)$, where $\nu_\alpha^*(w_\nu)$ is the unique root of $w'_\nu(1) = \alpha w_\nu(1)$.

The above equations are equivalent to

$$\begin{aligned} J_\nu(1)I_{\nu+1}(1) - J_{\nu+1}(1)I_\nu(1) + 2\nu(1-\alpha)J_\nu(1)I_\nu(1) &= 0, \\ J_\nu(1)I_{\nu+1}(1) - J_{\nu+1}(1)I_\nu(1) + (1-\alpha)J_\nu(1)I_\nu(1) &= 0 \end{aligned}$$

and

$$J_\nu(1)I_{\nu+1}(1) - J_{\nu+1}(1)I_\nu(1) + 4(1-\alpha)J_\nu(1)I_\nu(1) = 0.$$

□

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